Some topics on order preserving operator inequalities (Current topics on operator theory and operator inequalities)

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Some topics on order preserving operator inequalities

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Dedicated to the memory of Professor Tatsuo Noda in deep sorrow

Abstract. Furuta inequality asserts: *$A \geq B \geq 0$ ensures $A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ holds for all $r \geq 0$ and $p \geq 1$.* Inequalities of (a) **GFI type, I** and (b) **GFI type, II** are given via Furuta inequality. The following (1) and (2) are examples of (a), and (3) is an example of (b):

(1) *For $A, B > 0$, $A \geq B$ if and only if*

$$A^{1+r-t} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \leq 0$, $r \geq t$, $p \geq 1$ and $1 \geq s \geq \frac{1-t}{p-t}$.

(2) *For $A, B > 0$, If $A \geq B$ if and only if*

$$A^{1+r-t} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s \in [1,2]$.

(3) *For $A, B > 0$, $A \geq B$ if and only if*

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \leq -1$, $r \geq 0$, $p \geq 1$, and $\frac{2p-t+1}{p-t} \geq s \geq 1$.

We show that GFI is easily obtained by repeating (2).

§1 Introduction

In what follows, a capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. We write $A \gg B$ if $\log A \geq \log B$ for strictly positive operators A and B, which is called the chaotic order. It is known that $A \geq B > 0$ yields $A \gg B$ since $\log t$ is operator monotone.

We cite the following famous Löwner-Heinz inequality [20][18] established in 1934.

Theorem L-H (Löwner-Heinz inequality). *$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$*

As an extension of Theorem L-H, we obtained the following result.

Theorem F (Furuta inequality).

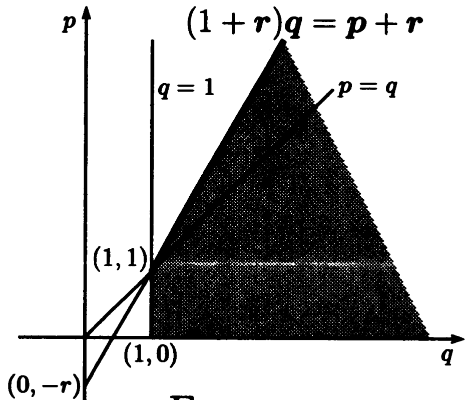
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

The original proof of Theorem F is given in [8], alternative proofs in [3],[19] and one page proof in [9]. The domain drawn for p,q and r in Figure is the best possible one for Theorem F in [21]. Next we state the following result which is an extension of Theorem F.

Theorem G (generalized Furuta inequality).
If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing of both r and s for any $r \geq t$ and $s \geq 1$, and the following inequality holds:

$$A^{1-t} = F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s),$$

that is,

(GFI) $$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $r \geq t$ and $s \geq 1$.

(GFI) interpolates Theorem F and the inequality equivalent to the main result of log majorization in [2]. The original proof of Theorem G is in [11], alternative proofs in [5],[17] and one page proof of (GFI) in [13]. The best possibility of (GFI) is obtained in [22], and [24][7].

Theorem FC ([4][10]). *The following* (i) *and* (ii) *hold:*

(i) *$A \gg B$ holds if and only if $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ holds for all $p \geq 0$ and $r \geq 0$.*

(ii) $\qquad A \gg B$ *holds if and only if* $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *holds for all* $p \geq 0$ *and* $r \geq 0$.

Theorem FC in the case $p = r$ is shown in [1], and Theorem FC can be regarded as Theorem F type inequality on chaotic order, and recently a breathtakingly elegant and simple proof of Theorem FC is given in [23] using only Theorem F. We posed in [16] that

(Q) $\qquad A \gg B$ *if and only if* $A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ *and* $s \geq 1$? And we gave a concrete counterexample to "only if" part of this question (Q) in [16]. To this question (Q), the following interesting answer is given in [6] by using their skilful method:

Theorem A. *For* $A, B > 0$, $A \geq B$ *if and only if*

$$A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ *and* $s \geq 1$.

Moreover the following affirmative answer to this question (Q) in some sense is given in [6]:

Theorem B. *For* $A, B > 0$, *If* $A \gg B$ (i.e., $\log A \geq \log B$) *if and only if*

$$A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for all $p \geq 0$, $r \geq 0$, $s \in [1,2]$ *and* $t \leq 0$.

We give several operator inequalities of two kinds of types associated with Theorem G and Theorem B by using only Theorem F and Theorem FC.

§2 GFI type operator inequalities

By using only Theorem F and Theorem FC, we show the following Theorem 2.1, Theorem 2.2 which are GFI type operator inequality, I and also we show Theorem 2.3 and Theorem 2.4 which are GFI type operator inequality, II.

(a) GFI type operator inequality, I

Firstly we state the following two theorems as GFI type operator inequality, I.

Theorem 2.1. *For* $A, B > 0$, $A \gg B$ (i.e., $\log A \geq \log B$) *if and only if*

$$A^{r-t} \geq (A^{\frac{r-t}{2}} B^{(p-t)s+t} A^{\frac{r-t}{2}})^{\frac{r-t}{(p-t)s+r}} \tag{2.1}$$

$$\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for all $t \leq 0$, $r \geq t$, $p \geq 0$ *and* $1 \geq s \geq \frac{-t}{p-t}$.

Theorem 2.2. *For* $A, B > 0$, $A \geq B$ *if and only if*

$$A^{1+r-t} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}} \tag{2.2}$$
$$\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \leq 0$, $r \geq t$, $p \geq 1$ *and* $1 \geq s \geq \frac{1-t}{p-t}$.

(b) GFI type operator inequality, II

Secondly we state the following two theorems of different type from (a).

Theorem 2.3.
(i) *For* $A, B > 0$, $A \gg B$ (i.e., $\log A \geq \log B$) *if and only if*

$$A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \tag{2.3}$$

holds for all $t \leq 0$, $r \geq 0$, $p \geq 0$, *and* $s \geq \frac{-t}{p-t}$.

(ii) *For* $A, B > 0$, $A \gg B$ (i.e., $\log A \geq \log B$) *if and only if*

$$A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \tag{2.3'}$$
$$\geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{r-t}{(p-t)s+r}}$$

holds for all $t \leq 0$, $r \geq 0$, $p \geq 0$, *and* $\frac{2p-t}{p-t} \geq s \geq 1$.

Theorem 2.4.
(i) *For* $A, B > 0$, $A \geq B$ *if and only if*

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{2.4}$$

holds for all $t \leq 0$, $r \geq 0$, $p \geq 1$, *and* $s \geq \frac{1-t}{p-t}$.

(ii) *For* $A, B > 0$, $A \geq B$ *if and only if*

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{2.4'}$$
$$\geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \leq -1$, $r \geq 0$, $p \geq 1$, *and* $\frac{2p-t+1}{p-t} \geq s \geq 1$.

First of all, to give proofs of our results, we state the following Theorem F_1, which is obtained by putting $q = \frac{p+r}{1+r} \geq 1$ for $p \geq 1$ and $r \geq 0$ in Theorem F.

Theorem F_1. *If $A \geq B \geq 0$, then the following* (i) *and* (ii) *hold:*

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \quad \textit{for all } p \geq 1 \textit{ and } r \geq 0$$

and

(ii) $$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \textit{for all } p \geq 1 \textit{ and } r \geq 0.$$

Next we state the following useful lemma to give proofs of our results.

Lemma A [11]. *Let X be a strictly positive operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

We omit almost all proofs of the results in this paper.

§3 Operator functions associated with Theorem 2.1 and Theorem 2.2

At first, as an application of Theorem 2.1, we state an operator function associated with Theorem 2.1, which is a parallel result to Theorem G.

Theorem 3.1. *Let $A \gg B$* (i.e.,$\log A \geq \log B$) *and let $G_{p,t}(A,B,r,s)$ be defined by*

$$G_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}A^{\frac{-r}{2}}$$

for real numbers p, r, s, and t. Then

(i). *In case $r \geq 0 \geq t$, $p \geq 0$ and $1 \geq s \geq \frac{-t}{p-t}$:*

$G_{p,t}(A,B,r,s)$ is decreasing of both r and s, and the following inequality holds:

$$A^{-t} = G_{p,t}(A,A,r,s) \tag{3.1}$$
$$\geq G_{p,t}(A,B,r,s).$$

(ii). *In case $0 \geq r \geq t$, $p \geq 0$ and $1 \geq s \geq \frac{-t}{p-t}$:*

$G_{p,t}(A,B,r,s)$ is decreasing of r and $G_{p,t}(A,B,r,s)$ is increasing of s, and the following inequality holds:

$$A^{-t} = G_{p,t}(A,A,r,s) \tag{3.2}$$
$$\geq G_{p,t}(A,B,r,s).$$

Next, as an application of Theorem 2.2, we state an operator function associated with Theorem 2.2, which is parallel results to Theorem 3.1 and Theorem G.

Theorem 3.2. *Let $A \geq B > 0$ and let $F_{p,t}(A,B,r,s)$ be defined by*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-r}{2}}$$

for real numbers $p,r,s,$ and t. Then

(i). *In case $r \geq 0 \geq t$, $p \geq 1$ and $1 \geq s \geq \frac{1-t}{p-t}$:*

$F_{p,t}(A,B,r,s)$ is decreasing of both r and s, and the following inequality holds :

$$\begin{aligned} A^{1-t} &= F_{p,t}(A,A,r,s) \\ &\geq F_{p,t}(A,B,r,s). \end{aligned} \tag{3.12}$$

(ii). *In case $0 \geq r \geq t$, $p \geq 1$ and $1 \geq s \geq \frac{1-t}{p-t}$:*

$F_{p,t}(A,B,r,s)$ is decreasing of r and $F_{p,t}(A,B,r,s)$ is increasing of s, and the following inequality holds:

$$\begin{aligned} A^{1-t} &= F_{p,t}(A,A,r,s) \\ &\geq F_{p,t}(A,B,r,s). \end{aligned} \tag{3.13}$$

§4 A result associated with Theorem G and Theorem 2.2, and a related counterexample and a conjecture

At first we state the following result which is quite similar to Theorem 2.2.

Proposition 4.1. *For $A,B > 0$, $A \geq B$ if and only if*

$$\begin{aligned} A^{1+r-t} &\geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}} \\ &\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \end{aligned} \tag{4.1}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s \in [1,2]$.

Proof. ($\Longleftarrow$) : We have only to put $r = t = 0$ in (4.1).

($\Longrightarrow$) : We recall $A^t \geq B^t$ by Löwner-Heinz theorem since $t \in [0,1]$, so we have $B^{-t} \geq A^{-t}$ by taking inverses of both sides. Then we obtain

$$\begin{aligned}
&\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \\
&= \{A^{\frac{r-t}{2}}B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}A^{\frac{r-t}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \qquad \text{by Lemma A} \\
&\leq \{A^{\frac{r-t}{2}}B^{\frac{p}{2}}(B^{\frac{p}{2}}B^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}A^{\frac{r-t}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \\
&= (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+t+r-t}} \\
&\leq A^{1+r-t}
\end{aligned}$$

and the first inequality follows by $B^{-t} \geq A^{-t}$ and Löwner-Heinz theorem since $s-1 \in [0,1]$ and $\frac{1+r-t}{(p-t)s+r} \in [0,1]$, and the second one follows by (ii) of Theorem F_1 since $(p-t)s+t \geq 1$ and $r-t \geq 0$, so we have (4.1). Whence the proof of Proposition 4.1 is complete.

Remark 4.1. Needless to say, it turns out that Proposition 4.1 belongs to GFI type operator inequality, I. Although the first inequality in (4.1) holds for *all* $s \geq 1$, $t \in [0,1]$, $r \geq t$ and $p \geq 1$ as seen in the proof of Proposition 4.1, the restricted condition $s \in [1,2]$ is required for the proof of the second inequality of (4.1).

Although Proposition 4.1 holds under the restricted condition $s \in [1,2]$, here we show a nice application of Proposition 4.1 as follows.

A simple proof of (GFI) by using Proposition 4.1

Proposition 4.1 asserts that $A \geq B > 0$ ensures

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{4.2}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s \in [1,2]$. In (4.2), put $A_1 = A^{1+r-t}$ and $B_1 = \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$. Then $A_1 \geq B_1 > 0$ by (4.2) holds, so by repeating (4.2), we have

$$A_1^{1+r_1-t_1} \geq \{A_1^{\frac{r_1}{2}}(A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}})^{s_1}A_1^{\frac{r_1}{2}}\}^{\frac{1+r_1-t_1}{(p_1-t_1)s_1+r_1}} \tag{4.3}$$

holds for all $t_1 \in [0,1]$, $r_1 \geq t_1$, $p_1 \geq 1$ and $s_1 \in [1,2]$. In (4.3), put p_1, r_1 and t_1 as follows:

$$p_1 = \frac{(p-t)s+r}{1+r-t} \geq 1 \qquad \text{and} \qquad r_1 = t_1 = \frac{r}{1+r-t} \in [0,1],$$

(4.3) ensures

$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{ss_1}A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)ss_1+r}} \tag{4.4}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $ss_1 \in [1,4]$. Repeating this process from (4.2) to (4.4), we obtain the desired inequality:

(GFI) $$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s \geq 1$. Whence the proof of (GFI) is complete.

Here we state a refinement of the proof of (GFI) in [11] and we remark that another simple proof of (GFI) is in [13].

Motivated by Proposition 4.1, Theorem 2.2 and parallelism between Theorem G and Theorem 2.2, we might apt to suppose the following question.

Question 4.1. *For $A, B > 0$, $A \geq B$ if and only if*

$$A^{1+r-t} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}} \tag{Q-4.1}$$
$$\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

holds for all $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s \geq 1$?.

But we have a counterexample to this Question 4.1, and we state a related conjecture.

A conjecture. *There exists strictly positive operators A and B such that $A \geq B > 0$ and*

$$(A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}} \not\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

for any $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s > 2$.

§5 Concluding remark

Remark 5.1. In what follows, let A and B be strictly positive operators. According to Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4, we define Type-I-(u), Type-I-(c), Type-II-(u) and Type-II-(c) as follows:

Type | **Content of type**

Type-I-(u) :
$$A^{1+r-t} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}}$$
$$\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

Type-I-(c) :
$$A^{r-t} \geq (A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}})^{\frac{r-t}{(p-t)s+r}}$$
$$\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

Type-II-(u) :
$$A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$$

$$\geq (A^{\frac{r-t}{2}} B^{(p-t)s+t} A^{\frac{r-t}{2}})^{\frac{1+r-t}{(p-t)s+r}}$$

Type-II-(c) :

$$A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

$$\geq (A^{\frac{r-t}{2}} B^{(p-t)s+t} A^{\frac{r-t}{2}})^{\frac{r-t}{(p-t)s+r}},$$

where Type-I-(u) means Type-I of the result on *the usual order* $A \geq B$ and Type-I-(c) means Type-I of the result on *the chaotic order* $\log A \geq \log B$ respectively and similarly Type-II-(u) means Type-II of the result on *the usual order* $A \geq B$ and Type-II-(c) means Type-II of the result on *the chaotic order* $\log A \geq \log B$ respectively.

We can enjoy an interesting contrast of the ranges of the parameters t, r, p and s, which clarifies Type-I-(u), Type-I-(c), Type-II-(u) and Type-II-(c) of the corresponding formulae.

Order	**Range**	**Type**	**Formula**
(i) $A \gg B$	$t \leq 0,\ r \geq t,\ p \geq 0,\ 1 \geq s \geq \frac{-t}{p-t}$	Type-I-(c)	(2.1) of Theorem 2.1
(ii) $A \geq B$	$t \leq 0,\ r \geq t,\ p \geq 1,\ 1 \geq s \geq \frac{1-t}{p-t}$	Type-I-(u)	(2.2) of Theorem 2.2
(iii) $A \geq B$	$t \in [0,1],\ r \geq t,\ p \geq 1,\ s \in [1,2]$	Type-I-(u)	(4.1) of Proposition 4.1
(iv) $A \gg B$	$t \leq 0,\ r \geq 0,\ p \geq 0,\ \frac{2p-t}{p-t} \geq s \geq 1$	Type-II-(c)	(2.3') of Theorem 2.3
(v) $A \geq B$	$t \leq -1,\ r \geq 0,\ p \geq 1,\ \frac{2p-t+1}{p-t} \geq s \geq 1$	Type-II-(u)	(2.4') of Theorem 2.4

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